Generate Very Large Sparse Matrices Starting from a Given Spectrum

Hervé Galicher[1], France Boillod-Cerneux[2,3,4], Serge Petiton[3,4] and Christophe Calvin[1]

[1] CEA Saclay DEN/DANS/DM2S, France
[2] LIFL, CNRS, Lille, France
[3] Université Lille 1, Sciences et Technologies, France
[4] Maison de la Simulation USR 3441 Bâtiment 565 - PC 190
CEA Saclay 91191 Gif-sur-Yvette cedex

Abstract. It is arduous to get matrices to evaluate the parallel performance and the behaviour of the numerical algorithms (linear and eigen solvers) at the extreme-scale. In the context of neutronic applications, we are particularly interested in eigenvalue problems: Numerical and scalability validation of frameworks require large matrices, ideally with known spectrum. We present in this paper a method to generate sparse matrices from a fixed spectrum and matching with some mathematical and shape properties.

1 The Sparse Matrix Generator

We propose to study a matrix generator which could produce huge size matrices, dense or sparse, non-Hermitian, with an imposed spectrum. We adress the following problem :

- the matrix spectrum must be the same as the choosen spectrum,
- the matrix shall not be Hermitian and shall not look like a trivial matrix,
- the algorithm must be tractable in very high dimension,
- the computer arithmetic must be controllable,
- the density and position of zeros must be controllable.

Such matrix generator aims to provide ideal matrices for numerical and parallel performances validation. Moreover, starting from a confidential matrix spectrum, we can generate a band-diagonal matrix with the same spectrum, avoiding many confidentiality locks.

Starting from a fixed spectrum $Spec_{in} \in \mathbb{C}^n$, we aim to generate a matrix $A_{gen} \in \mathbb{C}^{n\times n}$ such that $Spec(A_{gen}) = Spec_{in}$. The presented method is based on ODE's and nilpotent semigroup generators.

Theorem 1 *Let's consider a matrix $M \in \mathbb{C}^{n\times n}$. If M verifies :*

$$\frac{dM}{dt} = MA\text{-}AM,\ M(0) = M_0,$$

$$(2)$$

then the matrix M_t is similar to M_0. Considering this, M_t has the same eigenvalues than M_0.

We define by S the semi-groupe of the solution of equation 1 and its operator $\tilde{A}$ as follows:

$$\begin{cases} \tilde{A} : \mathbb{C}^{n\times n} \rightarrow \mathbb{C}^{n\times n} \\ \quad M \quad \rightarrow M.A - A.M, \end{cases}$$

We aim to find a matrix A that has relevant properties to simplify the computation and conserve $Spec(M_0)$. We impose $\tilde{A}$ to be a nilpotent operator, id est $\exists p \in \mathbb{N}$ such that $\tilde{A}^p = 0$. The $\tilde{A}$ nipoltency degree is related to A nipoltency degree itself. We denote by d the $\tilde{A}$ nipoltency degree ($d \in \mathbb{N}$), it has been prooved that the A nipoltency degree is equal to $\frac{d}{2}$. As A is nilpotent, $\exists B \in \mathbb{C}^{n\times n}$, invertible and $\exists P \in \mathbb{C}^{n\times n}$, such that $A = B^{-1}PB$.

To avoid the B^{-1} computation, we consider that $B = Id$. P is a diagonal block matrix and we denote by $P_k \in \mathbb{C}^{k\times k}$ the k^{th} diagonal block of P. $P_k \in \mathbb{C}^{k\times k}$ is a null matrix, except its sub-diagonal which has 1 value only and $\frac{d}{2} = \max_{i\in[1,k]}(size(P_i))$. By applying the exponential operator to the linear operator $\tilde{A}$ of S we obtain the following equation 3:

$$M(t) = e^{(\tilde{A}t)}.M_0 = M_0 + \tilde{A}t.M_0 + \tilde{A}^2.M_0\frac{t^2}{2!} + ... + \tilde{A}^d.M_0\frac{t^d}{d!},$$

Starting from the previous equation and a matrix M_0 whose subdiagonals are null and its spectrum is given by the user, we obtain a matrix M_t that matches with our hypothesis. A first parallel draft of the presented algorithm has been implemented. It has the advantage that, at the end, the final algorithm is surprisingly simple with parameters easily controllable to choose the shape of the final matrix. This algorithm allows us to produce matrices having a multidiagonal shape (beyond a controllable diagonal threshold, elements in the top-right and bottom-left corners are exactly zeros). A major aspect of this algorithm is the exact computation leading to an exact conservation of the matrix spectrum.

References

1. M. T Chu and S.-J. Wu: Matrix reconstruction with prescribed diagonal elements, eigenvalues, and singular values. Preprint, submitted, 2013.
2. M. T Chu and G. Golub: Structured Inverse Eigenvalue Problems. Acta Numerica, invited paper, (2002) 1–71
3. M. T Chu: A fast recursive algorithm for constructing matrices with prescribed eigenvalues and singular values. SIAM J. Numer. Anal. **37** (2000) 1004–1020
4. M. T Chu: On constructing matrices with prescribed singular values and diagonal elements. Linear Alg. Appl. **288** (1999) 11–22